
Contents

Preface	vii
1 Modular Forms, Elliptic Curves, and Modular Curves	1
1.1 First definitions and examples	1
1.2 Congruence subgroups	11
1.3 Complex tori	24
1.4 Complex tori as elliptic curves	31
1.5 Modular curves and moduli spaces	37
2 Modular Curves as Riemann Surfaces	45
2.1 Topology	45
2.2 Charts	48
2.3 Elliptic points	52
2.4 Cusps	57
2.5 Modular curves and Modularity	63
3 Dimension Formulas	65
3.1 The genus	65
3.2 Automorphic forms	71
3.3 Meromorphic differentials	77
3.4 Divisors and the Riemann–Roch Theorem	83
3.5 Dimension formulas for even k	85
3.6 Dimension formulas for odd k	89
3.7 More on elliptic points	92
3.8 More on cusps	98
3.9 More dimension formulas	106
4 Eisenstein Series	109
4.1 Eisenstein series for $\mathrm{SL}_2(\mathbf{Z})$	109
4.2 Eisenstein series for $\Gamma(N)$ when $k \geq 3$	111
4.3 Dirichlet characters, Gauss sums, and eigenspaces	116

4.4	Gamma, zeta, and L -functions	120
4.5	Eisenstein series for the eigenspaces when $k \geq 3$	126
4.6	Eisenstein series of weight 2	130
4.7	Bernoulli numbers and the Hurwitz zeta function	133
4.8	Eisenstein series of weight 1	138
4.9	The Fourier transform and the Mellin transform	143
4.10	Nonholomorphic Eisenstein series	147
4.11	Modular forms via theta functions	155
5	Hecke Operators	163
5.1	The double coset operator	163
5.2	The $\langle d \rangle$ and T_p operators	168
5.3	The $\langle n \rangle$ and T_n operators	178
5.4	The Petersson inner product	181
5.5	Adjoint of the Hecke Operators	183
5.6	Oldforms and Newforms	187
5.7	The Main Lemma	189
5.8	Eigenforms	195
5.9	The connection with L -functions	200
5.10	Functional equations	204
5.11	Eisenstein series again	205
6	Jacobians and Abelian Varieties	211
6.1	Integration, homology, the Jacobian, and Modularity	212
6.2	Maps between Jacobians	217
6.3	Modular Jacobians and Hecke operators	226
6.4	Algebraic numbers and algebraic integers	230
6.5	Algebraic eigenvalues	233
6.6	Eigenforms, Abelian varieties, and Modularity	240
7	Modular Curves as Algebraic Curves	249
7.1	Elliptic curves as algebraic curves	250
7.2	Algebraic curves and their function fields	257
7.3	Divisors on curves	268
7.4	The Weil pairing algebraically	275
7.5	Function fields over \mathbf{C}	279
7.6	Function fields over \mathbf{Q}	287
7.7	Modular curves as algebraic curves and Modularity	290
7.8	Isogenies algebraically	295
7.9	Hecke operators algebraically	300
8	The Eichler–Shimura Relation and L-functions	309
8.1	Elliptic curves in arbitrary characteristic	310
8.2	Algebraic curves in arbitrary characteristic	317
8.3	Elliptic curves over \mathbf{Q} and their reductions	322

8.4 Elliptic curves over $\overline{\mathbf{Q}}$ and their reductions 329

8.5 Reduction of algebraic curves and maps 336

8.6 Modular curves in characteristic p : Igusa’s Theorem 347

8.7 The Eichler–Shimura Relation 349

8.8 Fourier coefficients, L -functions, and Modularity 356

9 Galois Representations 365

9.1 Galois number fields 366

9.2 The ℓ -adic integers and the ℓ -adic numbers 372

9.3 Galois representations 376

9.4 Galois representations and elliptic curves 382

9.5 Galois representations and modular forms 386

9.6 Galois representations and Modularity 391

Hints and Answers to the Exercises 401

List of Symbols 421

Index 427

References 433