# Contents


**1 Geometry of complex numbers and quaternions** **1**

1.1 Rotations of the plane . . . . . 2
1.2 Matrix representation of complex numbers . . . . . 5
1.3 Quaternions . . . . . 7
1.4 Consequences of multiplicative absolute value . . . . . 11
1.5 Quaternion representation of space rotations . . . . . 14
1.6 Discussion . . . . . 18

**2 Groups** **23**

2.1 Crash course on groups . . . . . 24
2.2 Crash course on homomorphisms . . . . . 27
2.3 The groups SU(2) and SO(3) . . . . . 32
2.4 Isometries of $\mathbb{R}^n$ and reflections . . . . . 36
2.5 Rotations of $\mathbb{R}^4$ and pairs of quaternions . . . . . 38
2.6 Direct products of groups . . . . . 40
2.7 The map from SU(2)$\times$SU(2) to SO(4) . . . . . 42
2.8 Discussion . . . . . 45

**3 Generalized rotation groups** **48**

3.1 Rotations as orthogonal transformations . . . . . 49
3.2 The orthogonal and special orthogonal groups . . . . . 51
3.3 The unitary groups . . . . . 54
3.4 The symplectic groups . . . . . 57
3.5 Maximal tori and centers . . . . . 60
3.6 Maximal tori in SO($n$), U($n$), SU($n$), Sp($n$) . . . . . 62
3.7 Centers of SO($n$), U($n$), SU($n$), Sp($n$) . . . . . 67
3.8 Connectedness and discreteness . . . . . 69
3.9 Discussion . . . . . 71

**4 The exponential map** **74**
4.1 The exponential map onto SO(2) . . . . . 75
4.2 The exponential map onto SU(2) . . . . . 77
4.3 The tangent space of SU(2) . . . . . 79
4.4 The Lie algebra $\mathfrak{su}(2)$ of SU(2) . . . . . 82
4.5 The exponential of a square matrix . . . . . 84
4.6 The affine group of the line . . . . . 87
4.7 Discussion . . . . . 91

**5 The tangent space** **93**
5.1 Tangent vectors of O($n$), U($n$), Sp($n$) . . . . . 94
5.2 The tangent space of SO($n$) . . . . . 96
5.3 The tangent space of U($n$), SU($n$), Sp($n$) . . . . . 99
5.4 Algebraic properties of the tangent space . . . . . 103
5.5 Dimension of Lie algebras . . . . . 106
5.6 Complexification . . . . . 107
5.7 Quaternion Lie algebras . . . . . 111
5.8 Discussion . . . . . 113

**6 Structure of Lie algebras** **116**
6.1 Normal subgroups and ideals . . . . . 117
6.2 Ideals and homomorphisms . . . . . 120
6.3 Classical non-simple Lie algebras . . . . . 122
6.4 Simplicity of $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{su}(n)$ . . . . . 124
6.5 Simplicity of $\mathfrak{so}(n)$ for $n > 4$ . . . . . 127
6.6 Simplicity of $\mathfrak{sp}(n)$ . . . . . 133
6.7 Discussion . . . . . 137

**7 The matrix logarithm** **139**
7.1 Logarithm and exponential . . . . . 140
7.2 The exp function on the tangent space . . . . . 142
7.3 Limit properties of log and exp . . . . . 145
7.4 The log function into the tangent space . . . . . 147
7.5 SO($n$), SU($n$), and Sp($n$) revisited . . . . . 150
7.6 The Campbell–Baker–Hausdorff theorem . . . . . 152
7.7 Eichler's proof of Campbell–Baker–Hausdorff . . . . . 154
7.8 Discussion . . . . . 158

**8 Topology** **160**
8.1 Open and closed sets in Euclidean space . . . . . 161
8.2 Closed matrix groups . . . . . 164
8.3 Continuous functions . . . . . 166
8.4 Compact sets . . . . . 169
8.5 Continuous functions and compactness . . . . . 171
8.6 Paths and path-connectedness . . . . . 173
8.7 Simple connectedness . . . . . 177
8.8 Discussion . . . . . 182

**9 Simply connected Lie groups** **186**
9.1 Three groups with tangent space $\mathbb{R}$ . . . . . 187
9.2 Three groups with the cross-product Lie algebra . . . . . 188
9.3 Lie homomorphisms . . . . . 191
9.4 Uniform continuity of paths and deformations . . . . . 194
9.5 Deforming a path in a sequence of small steps . . . . . 195
9.6 Lifting a Lie algebra homomorphism . . . . . 197
9.7 Discussion . . . . . 201

**Bibliography** **204**

**Index** **207**