

# Contents

<b>Preface</b>	<b>vii</b>
<b>Introduction</b>	<b>1</b>
<b>I Numbers and Functions, Sequences and Limits</b>	<b>5</b>
<b>1 Mathematical Modeling</b>	<b>7</b>
1.1 The Dinner Soup Model . . . . .	8
1.2 The Muddy Yard Model . . . . .	10
1.3 Mathematical Modeling . . . . .	11
<b>2 Natural Numbers Just Aren't Enough</b>	<b>15</b>
2.1 The Natural Numbers . . . . .	15
2.2 Infinity or Is There a Largest Natural Number? . . . . .	18
2.3 A Controversy About the Set of Natural Numbers . . . . .	19
2.4 Subtraction and the Integers . . . . .	21
2.5 Division and the Rational Numbers . . . . .	23
2.6 Distance and the Absolute Value . . . . .	24
2.7 Computer Representation of Integers . . . . .	25
<b>3 Infinity and Mathematical Induction</b>	<b>29</b>
3.1 The Need for Induction . . . . .	29
3.2 The Principle of Mathematical Induction . . . . .	31

3.3	Using Induction . . . . .	32
3.4	A Model of an Insect Population . . . . .	33
<b>4</b>	<b>Rational Numbers</b>	<b>37</b>
4.1	Operating with Rational Numbers . . . . .	38
4.2	Decimal Expansions of Rational Numbers . . . . .	40
4.3	The Set of Rational Numbers . . . . .	45
4.4	The Verhulst Model of Populations . . . . .	45
4.5	A Model of Chemical Equilibrium . . . . .	46
4.6	The Rational Number Line . . . . .	47
<b>5</b>	<b>Functions</b>	<b>51</b>
5.1	Functions . . . . .	51
5.2	Functions and Sets . . . . .	53
5.3	Graphing Functions of Integers . . . . .	55
5.4	Graphing Functions of Rational Numbers . . . . .	58
<b>6</b>	<b>Polynomials</b>	<b>61</b>
6.1	Polynomials . . . . .	61
6.2	The $\Sigma$ Notation for Sums . . . . .	62
6.3	Arithmetic with Polynomials . . . . .	63
6.4	Equality of Polynomials . . . . .	67
6.5	Graphs of Polynomials . . . . .	68
6.6	Piecewise Polynomial Functions . . . . .	68
<b>7</b>	<b>Functions, Functions, and More Functions</b>	<b>73</b>
7.1	Linear Combinations of Functions . . . . .	73
7.2	Multiplication and Division of Functions . . . . .	76
7.3	Rational Functions . . . . .	78
7.4	Composition of Functions . . . . .	79
<b>8</b>	<b>Lipschitz Continuity</b>	<b>83</b>
8.1	Continuous Behavior and Linear Functions . . . . .	83
8.2	The Definition of Lipschitz Continuity . . . . .	85
8.3	Bounded Sets of Numbers . . . . .	88
8.4	Monomials . . . . .	89
8.5	Linear Combinations of Functions . . . . .	92
8.6	Bounded Functions . . . . .	93
8.7	Products and Quotients of Functions . . . . .	94
8.8	The Composition of Functions . . . . .	96
<b>9</b>	<b>Sequences and Limits</b>	<b>99</b>
9.1	The First Encounter with Sequences and Limits . . . . .	99
9.2	The Mathematical Definition of a Limit . . . . .	101
9.3	Some Background on the Definition of a Limit . . . . .	106

9.4	Divergent Sequences . . . . .	107
9.5	Infinite Series . . . . .	108
9.6	Limits Are Unique . . . . .	110
9.7	Arithmetic with Sequences . . . . .	111
9.8	Functions and Sequences . . . . .	113
9.9	Sequences with Rational Elements . . . . .	116
9.10	Calculus and Computing Limits . . . . .	117
9.11	Computer Representation of Rational Numbers . . . . .	118
<b>10</b>	<b>Solving the Muddy Yard Model</b>	<b>125</b>
10.1	Rational Numbers Just Aren't Enough . . . . .	125
10.2	Infinite Nonperiodic Decimal Expansions . . . . .	128
10.3	The Bisection Algorithm for the Muddy Yard Model . . . . .	129
10.4	The Bisection Algorithm Converges . . . . .	131
10.5	... and the Limit Solves the Muddy Yard Model . . . . .	132
<b>11</b>	<b>Real Numbers</b>	<b>135</b>
11.1	Irrational Numbers . . . . .	136
11.2	Arithmetic with Irrational Numbers . . . . .	138
11.3	Inequalities for Irrational Numbers . . . . .	141
11.4	The Real Numbers . . . . .	143
11.5	Please Oh Please, Let the Real Numbers Be Enough . . . . .	143
11.6	Some History of the Real Numbers . . . . .	148
<b>12</b>	<b>Functions of Real Numbers</b>	<b>153</b>
12.1	Functions of a Real Variable . . . . .	153
12.2	Extending Functions of Rational Numbers . . . . .	154
12.3	Graphing Functions of a Real Variable . . . . .	156
12.4	Limits of a Function of a Real Variable . . . . .	158
<b>13</b>	<b>The Bisection Algorithm</b>	<b>165</b>
13.1	The Bisection Algorithm for General Root Problems . . . . .	165
13.2	Solving the Model of Chemical Equilibrium . . . . .	166
13.3	The Bisection Algorithm Converges . . . . .	168
13.4	When to Stop the Bisection Algorithm . . . . .	170
13.5	Power Functions . . . . .	171
13.6	Computing Roots by the Decasection Algorithm . . . . .	172
<b>14</b>	<b>Inverse Functions</b>	<b>179</b>
14.1	A Geometric Investigation . . . . .	179
14.2	An Analytic Investigation . . . . .	183
<b>15</b>	<b>Fixed Points and Contraction Maps</b>	<b>191</b>
15.1	The Greeting Card Sales Model . . . . .	192
15.2	The Free Time Model . . . . .	193

15.3	Fixed Point Problems and Root Problems . . . . .	194
15.4	Solving the Greeting Card Sales Model . . . . .	197
15.5	The Fixed Point Iteration . . . . .	200
15.6	Convergence of the Fixed Point Iteration . . . . .	201
15.7	Rates of Convergence . . . . .	206
<b>II Differential and Integral Calculus</b>		<b>215</b>
<b>16</b>	<b>The Linearization of a Function at a Point</b>	<b>217</b>
16.1	The Imprecision of Lipschitz Continuity . . . . .	217
16.2	Linearization at a Point . . . . .	221
16.3	A Systematic Approach . . . . .	224
16.4	Strong Differentiability and Smoothness . . . . .	228
<b>17</b>	<b>Analyzing the Behavior of a Population Model</b>	<b>231</b>
17.1	A General Population Model . . . . .	231
17.2	Equilibrium Points and Stability . . . . .	232
<b>18</b>	<b>Interpretations of the Derivative</b>	<b>237</b>
18.1	A Geometric Picture . . . . .	237
18.2	Rates of Change . . . . .	240
18.3	Differentiability and Strong Differentiability . . . . .	242
<b>19</b>	<b>Differentiability on Intervals</b>	<b>245</b>
19.1	Strong Differentiability on Intervals . . . . .	245
19.2	Uniform Strong Differentiability . . . . .	250
19.3	Uniform Strong Differentiability and Smoothness . . . . .	251
19.4	Closed Intervals and One-Sided Linearization . . . . .	253
19.5	Differentiability on Intervals . . . . .	256
<b>20</b>	<b>Useful Properties of the Derivative</b>	<b>259</b>
20.1	Linear Combinations of Functions . . . . .	259
20.2	Products of Functions . . . . .	261
20.3	Composition of Functions . . . . .	263
20.4	Quotients of Functions . . . . .	265
20.5	Derivatives of Derivatives: Descent into Despair . . . . .	266
<b>21</b>	<b>The Mean Value Theorem</b>	<b>269</b>
21.1	A Constructive Proof . . . . .	271
21.2	An Application to Monotonicity . . . . .	276
<b>22</b>	<b>Derivatives of Inverse Functions</b>	<b>279</b>
22.1	The Lipschitz Continuity of an Inverse Function . . . . .	279
22.2	The Differentiability of an Inverse Function . . . . .	281

<b>23 Modeling with Differential Equations</b>	<b>285</b>
23.1 Newton's Law of Motion . . . . .	286
23.2 Einstein's Law of Motion . . . . .	288
23.3 Describing Differential Equations . . . . .	288
23.4 Solutions of Differential Equations . . . . .	290
23.5 Uniqueness of Solutions . . . . .	292
23.6 Solving Galileo's Model of a Free-Falling Object . . . . .	296
<b>24 Antidifferentiation</b>	<b>301</b>
24.1 Antidifferentiation . . . . .	302
24.2 The Indefinite Integral . . . . .	302
24.3 Sophisticated Guesswork . . . . .	303
24.4 The Method of Substitution . . . . .	305
24.5 The Language of Differentials . . . . .	307
24.6 The Method of Integration by Parts . . . . .	309
24.7 Definite Integrals . . . . .	310
<b>25 Integration</b>	<b>315</b>
25.1 A Simple Case . . . . .	316
25.2 A First Attempt at Approximation . . . . .	317
25.3 Approximating the Solution on a Large Interval . . . . .	318
25.4 Uniform Cauchy Sequences of Functions . . . . .	323
25.5 Convergence of the Integration Approximation . . . . .	327
25.6 The Limit Solves the Differential Equation . . . . .	331
25.7 The Fundamental Theorem of Calculus . . . . .	333
<b>26 Properties of the Integral</b>	<b>339</b>
26.1 Linearity . . . . .	339
26.2 Monotonicity . . . . .	340
26.3 Playing with the Limits . . . . .	341
26.4 More on Definite and Indefinite Integrals . . . . .	343
<b>27 Applications of the Integral</b>	<b>345</b>
27.1 Area Under a Curve . . . . .	346
27.2 Average Value of a Function . . . . .	351
<b>28 Rocket Propulsion and the Logarithm</b>	<b>355</b>
28.1 A Model of Rocket Propulsion . . . . .	355
28.2 The Definition and Graph of the Logarithm . . . . .	358
28.3 Two Important Properties of the Logarithm . . . . .	359
28.4 Irrational Exponents . . . . .	361
28.5 Power Functions . . . . .	362
28.6 Change of Base . . . . .	363
28.7 Solving the Model of Rocket Propulsion . . . . .	364
28.8 Derivatives and Integrals Involving the Logarithm . . . . .	365

<b>29 Constant Relative Rate of Change and the Exponential</b>	<b>369</b>
29.1 Models Involving a Constant Relative Rate of Change . . .	369
29.2 The Exponential Function . . . . .	371
29.3 Solution of the Model for Constant Relative Rate of Change . . . . .	374
29.4 More on Integrating Factors . . . . .	375
29.5 General Exponential Functions . . . . .	377
29.6 Rates of Growth of the Exponential and Logarithm . . . . .	379
29.7 Justification of the Continuous Model . . . . .	380
<b>30 A Mass-Spring System and the Trigonometric Functions</b>	<b>387</b>
30.1 Hooke's Model of a Mass-Spring System . . . . .	387
30.2 The Smoothness of Trigonometric Functions . . . . .	389
30.3 Solving the Model for a Mass-Spring System . . . . .	394
30.4 Inverse Trigonometric Functions . . . . .	396
<b>31 Fixed Point Iteration and Newton's Method</b>	<b>407</b>
31.1 Linearization and the Fixed Point Iteration . . . . .	407
31.2 Global Convergence and Local Behavior . . . . .	408
31.3 High Order Convergence . . . . .	414
31.4 Newton's Method . . . . .	417
31.5 Some Interpretations and History of Newton's Method . . .	420
31.6 What Is the Error in an Approximate Root? . . . . .	422
31.7 Globally Convergent Methods . . . . .	424
31.8 When Good Derivatives Are Hard to Find . . . . .	426
31.9 Unanswered Questions . . . . .	429
<b>Calculus Quagmires</b>	<b>435</b>
<b>III You Want Analysis? We've Got Your Analysis Right Here.</b>	<b>439</b>
<b>32 Notions of Continuity and Differentiability</b>	<b>441</b>
32.1 A General Notion of Continuity . . . . .	441
32.2 Properties of Continuous Functions . . . . .	443
32.3 Continuity on an Interval . . . . .	443
32.4 Differentiability and Strong Differentiability . . . . .	448
32.5 Weierstrass' Principle and Uniform Continuity . . . . .	451
32.6 Some Differentiability Equivalences . . . . .	457
<b>33 Sequences of Functions</b>	<b>463</b>
33.1 Uniform Convergence and Continuity . . . . .	465

33.2	Uniform Convergence and Differentiability . . . . .	467
33.3	Uniform Convergence and Integrability . . . . .	471
33.4	Unanswered Questions . . . . .	472
<b>34</b>	<b>Relaxing Integration</b>	<b>477</b>
34.1	Continuous Functions . . . . .	477
34.2	General Meshes . . . . .	482
34.3	Application to Computing the Length of a Curve . . . . .	487
<b>35</b>	<b>Delicate Limits and Gross Behavior</b>	<b>493</b>
35.1	Functions and Infinity . . . . .	493
35.2	L'Hôpital's Rule . . . . .	496
35.3	Orders of Magnitude . . . . .	502
<b>36</b>	<b>The Weierstrass Approximation Theorem</b>	<b>509</b>
36.1	The Binomial Expansion . . . . .	510
36.2	The Law of Large Numbers . . . . .	513
36.3	The Modulus of Continuity . . . . .	516
36.4	The Bernstein Polynomials . . . . .	517
36.5	Accuracy and Convergence . . . . .	521
36.6	Unanswered Questions . . . . .	522
<b>37</b>	<b>The Taylor Polynomial</b>	<b>525</b>
37.1	A Quadratic Approximation . . . . .	525
37.2	Taylor's Representation of a Polynomial . . . . .	526
37.3	The Taylor Polynomial for a General Function . . . . .	528
37.4	The Error of the Taylor Polynomial . . . . .	530
37.5	Another Point of View . . . . .	534
37.6	Accuracy and Convergence . . . . .	535
37.7	Unanswered Questions . . . . .	538
37.8	Some History of Taylor Polynomials . . . . .	539
<b>38</b>	<b>Polynomial Interpolation</b>	<b>543</b>
38.1	Existence and Uniqueness . . . . .	544
38.2	The Error of a Polynomial Interpolant . . . . .	548
38.3	Accuracy and Convergence . . . . .	550
38.4	A Piecewise Polynomial Interpolant . . . . .	553
38.5	Unanswered Questions . . . . .	555
<b>39</b>	<b>Nonlinear Differential Equations</b>	<b>559</b>
39.1	A Warning . . . . .	566
<b>40</b>	<b>The Picard Iteration</b>	<b>569</b>
40.1	Operators and Spaces of Functions . . . . .	570
40.2	A Fixed Point Problem for a Differential Equation . . . . .	571
40.3	The Banach Contraction Mapping Principle . . . . .	573

40.4 Picard's Iteration . . . . .	575
40.5 Unanswered Questions . . . . .	578
<b>41 The Forward Euler Method</b>	<b>583</b>
41.1 The Forward Euler Method . . . . .	583
41.2 Equicontinuity and Arzela's Theorem . . . . .	586
41.3 Convergence of Euler's Method . . . . .	592
41.4 Uniqueness and Continuous Dependence on Initial Data . .	596
41.5 More on the Convergence of Euler's Method . . . . .	598
41.6 Unanswered Questions . . . . .	599
<b>A Conclusion or an Introduction?</b>	<b>605</b>
<b>References</b>	<b>607</b>
<b>Index</b>	<b>609</b>