
Contents

1	Preliminaries	1
1.1	Hausdorff and Minkowski dimensions	1
1.2	The area and coarea formulae	3
1.3	Approximation numbers	6
1.4	Inequalities	9
2	Hardy-type Operators	11
2.1	Introduction	11
2.2	Boundedness of T	12
2.3	Compactness of T	17
2.4	Approximation numbers of T	23
2.4.1	The Hardy operator on a finite interval	24
2.4.2	The general case: Preliminaries	31
2.4.3	Estimates for $a_m(T)$, $1 < p \leq q < \infty$	39
2.4.4	Estimates for $a_n(T)$ when $p = 1$ or $q = \infty$	42
2.4.5	Approximation numbers of T when $1 \leq q < p \leq \infty$	43
2.4.6	Asymptotic results for $p = q \in (1, \infty)$	43
2.4.7	The cases $p = 1, \infty$	50
2.5	l_α and $l_{\alpha,w}$ classes.	51
2.6	Hardy-type operators on trees	55
2.6.1	Analysis on trees	55
2.6.2	Boundedness of T	57
2.7	Compactness of T and its approximation numbers	58
2.8	Notes	59
3	Banach function spaces	63
3.1	Introduction	63
3.1.1	Definitions	64
3.2	Rearrangements	69
3.3	Rearrangement-invariant spaces	84
3.4	Examples	90

3.4.1	Lorentz, Lorentz-Zygmund and generalised Lorentz-Zygmund spaces	90
3.4.2	Orlicz spaces	96
3.4.3	Lorentz-Karamata spaces	108
3.4.4	Decompositions	121
3.5	Operators of joint weak type	125
3.5.1	Definitions	125
3.5.2	Operators of strong and weak type	128
3.6	Bessel-Lorentz-Karamata-potential spaces	133
3.6.1	Abstract Sobolev spaces	133
3.6.2	Bessel-Lorentz-Karamata-potential spaces	134
3.6.3	Sub-limiting embeddings	139
3.6.4	Limiting embeddings	140
3.6.5	Super-limiting embeddings	144
3.7	Examples	152
3.8	Other spaces	155
3.9	Notes	158
4	Poincaré and Hardy inequalities	161
4.1	Introduction	161
4.2	Poincaré inequalities in BFSs	164
4.2.1	Poincaré and Friedrichs inequalities	164
4.2.2	Examples	174
4.2.3	Higher-order cases	183
4.3	Concrete spaces	185
4.3.1	Classes of domains	185
4.3.2	Sobolev and Poincaré inequalities	193
4.4	Hardy inequalities	207
4.5	Notes	217
5	Generalised ridged domains	219
5.1	Introduction	219
5.1.1	Ridges and skeletons	220
5.1.2	Simple ridges in \mathbb{R}^2	224
5.2	Generalised ridged domains	228
5.3	Measure of non-compactness	234
5.4	Analysis on GRD	244
5.4.1	The map T and its approximate inverse M	245
5.4.2	Equivalent embeddings	249
5.4.3	Equivalent Poincaré inequalities	251
5.5	Compactness of E	252
5.5.1	Local compactness	252
5.5.2	Measure of non-compactness	254
5.6	Embedding Theorems	261

5.7	The Poincaré inequality and $\alpha(E)$	266
5.8	Notes	273
6	Approximation numbers of Sobolev embeddings	275
6.1	Introduction	275
6.2	Some quotient space norms	277
6.3	Dirichlet-Neumann bracketing in L_p	282
6.4	Further asymptotic estimates for a GRD Ω	294
6.5	Notes	305
	References	307
	Author Index	319
	Subject Index	323
	Notation Index	325