Contents

Introductory Remarks on Computer Algebra 1

1 Basic Notations and Ideas: A Historical Account 11

2 Basic Computational Problems and Their Solution 37
2.1 The Geometry-Algebra Dictionary 37
2.2 Basic Applications of Gröbner Bases 51

3 An Introduction to SINGULAR 63
3.1 General Remarks on SINGULAR and its Syntax 63
3.2 Rings in SINGULAR 66
3.2.1 Global Monomial Orders 68
3.2.2 Creating Ring Maps 72
3.3 Ideals, Vectors and Modules in SINGULAR 74
3.4 Handling Graded Modules 76
3.5 Computing Gröbner Bases 78
3.6 Basic Applications of Gröbner Bases (revisited) 87
3.6.1 Ideal Membership Test 87
3.6.2 Elimination 89
3.6.3 Kernel of a Ring Map 91
3.6.4 Test for Subalgebra Membership 93
3.6.5 Test for Surjectivity of a Ring Map 94
3.6.6 Syzygies and Free Resolutions 94
3.7 Gröbner Bases over Noncommutative Algebras 103
3.8 Writing SINGULAR Procedures and Libraries 111
3.9 Communication with Other Systems 122
3.10 Visualization: Plotting Curves and Surfaces 123

Practical Session I 125

Practical Session II 127

4 Homological Algebra I . . . 129
4.1 Lifting Homomorphisms . . . 129
4.2 Constructive Module Theory . . . 130
4.2.1 Cokernels and Mapping Cones . . . 131
4.2.2 Modulo . . . 131
4.2.3 Kernel, Hom, Ext, Tor, and more . . . 132
4.2.4 Some Explicit Constructions . . . 137

5 Homological Algebra II . . . 145
5.1 Flatness . . . 145
5.2 Depth and Codimension . . . 156
5.3 Cohen-Macaulay Rings . . . 161

Practical Session III . . . 167

6 Solving Systems of Polynomial Equations . . . 169
6.1 Gröbner Basis Techniques . . . 170
6.1.1 Computing Dimension . . . 170
6.1.2 Zero-Dimensional Solving by Elimination . . . 172
6.1.3 Decomposition . . . 173
6.2 Resultant Based Methods . . . 183
6.2.1 The Sylvester Resultant . . . 183
6.2.2 Multipolynomial Resultants . . . 188
6.2.3 Zero-Dimensional Solving via Resultants . . . 190

7 Primary Decomposition and Normalization . . . 201
7.1 Primary Decomposition . . . 201
7.2 Normalization . . . 210

Practical Session IV . . . 217

8 Algorithms for Invariant Theory . . . 219
8.1 Finite Groups . . . 220
8.1.1 The Nonmodular Case . . . 220
8.1.2 The Modular Case . . . 229
8.1.3 Quotients for Finite Group Actions . . . 231
8.2 Linearly Reductive Groups . . . 233

9 Computing in Local Rings . . . 235
9.1 Rings Implemented by Monomial Orders . . . 238
9.2 Standard Bases and their Computation . . . 242
9.3 Factorization and Primary Decomposition . . . 248
9.4 Computing Dimension . . . 249
9.5 Elimination . . . 261
9.6 Hamburger-Noether Expansion . . . 264

Practical Session V 271

Appendix A Sheaf Cohomology and Beilinson Monads 273

Appendix B Solutions to Exercises 281

References 311

Index 319