

# Contents

Preface	vii
Comments on the Use of This Book	x
<b>PART I: LARGE DEVIATIONS AND STATISTICAL MECHANICS</b>	
<b>Chapter I. Introduction to Large Deviations</b>	<b>3</b>
I.1. Overview	3
I.2. Large Deviations for I.I.D. Random Variables with a Finite State Space	9
I.3. Levels-1 and 2 for Coin Tossing	11
I.4. Levels-1 and 2 for I.I.D. Random Variables with a Finite State Space	13
I.5. Level-3: Empirical Pair Measure	17
I.6. Level-3: Empirical Process	21
I.7. Notes	26
I.8. Problems	27
<b>Chapter II. Large Deviation Property and Asymptotics of Integrals</b>	<b>30</b>
II.1. Introduction	30
II.2. Levels-1, 2, and 3 Large Deviations for I.I.D. Random Vectors	30
II.3. The Definition of Large Deviation Property	33
II.4. Statement of Large Deviation Properties for Levels-1, 2, and 3	37
II.5. Contraction Principles	42
II.6. Large Deviation Property for Random Vectors and Exponential Convergence	46
II.7. Varadhan's Theorem on the Asymptotics of Integrals	50
II.8. Notes	55
II.9. Problems	56
<b>Chapter III. Large Deviations and the Discrete Ideal Gas</b>	<b>59</b>
III.1. Introduction	59
III.2. Physics Prelude: Thermodynamics	60
III.3. The Discrete Ideal Gas and the Microcanonical Ensemble	64

III.4.	Thermodynamic Limit, Exponential Convergence, and Equilibrium Values	66
III.5.	The Maxwell–Boltzmann Distribution and Temperature	74
III.6.	The Canonical Ensemble and Its Equivalence with the Microcanonical Ensemble	75
III.7.	A Derivation of a Thermodynamic Equation	78
III.8.	The Gibbs Variational Formula and Principle	79
III.9.	Notes	84
III.10.	Problems	86
Chapter IV. Ferromagnetic Models on $\mathbb{Z}$		88
IV.1.	Introduction	88
IV.2.	An Overview of Ferromagnetic Models	88
IV.3.	Finite-Volume Gibbs States on $\mathbb{Z}$	94
IV.4.	Spontaneous Magnetization for the Curie–Weiss Model	98
IV.5.	Spontaneous Magnetization for General Ferromagnets on $\mathbb{Z}$	103
IV.6.	Infinite-Volume Gibbs States and Phase Transitions	109
IV.7.	The Gibbs Variational Formula and Principle	125
IV.8.	Notes	131
IV.9.	Problems	134
Chapter V. Magnetic Models on $\mathbb{Z}^D$ and on the Circle		138
V.1.	Introduction	138
V.2.	Finite-Volume Gibbs States on $\mathbb{Z}^D$ , $D \geq 1$	140
V.3.	Moment Inequalities	142
V.4.	Properties of the Magnetization and the Gibbs Free Energy	146
V.5.	Spontaneous Magnetization on $\mathbb{Z}^D$ , $D \geq 2$ , Via the Peierls Argument	153
V.6.	Infinite-Volume Gibbs States and Phase Transitions	157
V.7.	Infinite-Volume Gibbs States and the Central Limit Theorem	162
V.8.	Critical Phenomena and the Breakdown of the Central Limit Theorem	170
V.9.	Three Faces of the Curie–Weiss Model	179
V.10.	The Circle Model and Random Waves	190
V.11.	A Postscript on Magnetic Models	198
V.12.	Notes	199
V.13.	Problems	203
PART II: CONVEXITY AND PROOFS OF LARGE DEVIATION THEOREMS		
Chapter VI. Convex Functions and the Legendre–Fenchel Transform		211
VI.1.	Introduction	211
VI.2.	Basic Definitions	211
VI.3.	Properties of Convex Functions	213

Contents	xiii
VI.4. A One-Dimensional Example of the Legendre–Fenchel Transform	218
VI.5. The Legendre–Fenchel Transform for Convex Functions on $\mathbb{R}^d$	220
VI.6. Notes	225
VI.7. Problems	226
Chapter VII. Large Deviations for Random Vectors	229
VII.1. Statement of Results	229
VII.2. Properties of $I_{\mathcal{W}}$	231
VII.3. Proof of the Large Deviation Bounds for $d = 1$	232
VII.4. Proof of the Large Deviation Bounds for $d \geq 1$	235
VII.5. Level-1 Large Deviations for I.I.D. Random Vectors	238
VII.6. Exponential Convergence and Proof of Theorem II.6.3	242
VII.7. Notes	243
VII.8. Problems	245
Chapter VIII. Level-2 Large Deviations for I.I.D. Random Vectors	250
VIII.1. Introduction	250
VIII.2. The Level-2 Large Deviation Theorem	250
VIII.3. The Contraction Principle Relating Levels-1 and 2 ( $d = 1$ )	253
VIII.4. The Contraction Principle Relating Levels-1 and 2 ( $d \geq 2$ )	258
VIII.5. Notes	264
VIII.6. Problems	265
Chapter IX. Level-3 Large Deviations for I.I.D. Random Vectors	269
IX.1. Statement of Results	269
IX.2. Properties of the Level-3 Entropy Function	270
IX.3. Contraction Principles	276
IX.4. Proof of the Level-3 Large Deviation Bounds	279
IX.5. Notes	287
IX.6. Problems	288
APPENDICES	
Appendix A: Probability	295
A.1. Introduction	295
A.2. Measurability	295
A.3. Product Spaces	296
A.4. Probability Measures and Expectation	297
A.5. Convergence of Random Vectors	299
A.6. Conditional Expectation, Conditional Probability, and Regular Conditional Distribution	299
A.7. The Kolmogorov Existence Theorem	301

A.8.	Weak Convergence of Probability Measures on a Metric Space	303
A.9.	The Space $\mathcal{M}_s((\mathbb{R}^d)^{\mathbb{Z}})$ and the Ergodic Theorem	306
A.10.	$n$ -Dependent Markov Chains	311
A.11.	Probability Measures on the Space $\{1, -1\}^{\mathbb{Z}^D}$	312
Appendix B: Proofs of Two Theorems in Section II.7		319
B.1.	Proof of Theorem II.7.1	319
B.2.	Proof of Theorem II.7.2	321
Appendix C: Equivalent Notions of Infinite-Volume Measures for Spin Systems		323
C.1.	Introduction	323
C.2.	Two-Body Interactions and Infinite-Volume Gibbs States	323
C.3.	Many-Body Interactions and Infinite-Volume Gibbs States	325
C.4.	DLR States	326
C.5.	The Gibbs Variational Formula and Principle	327
C.6.	Solution of the Gibbs Variational Formula for Finite-Range Interactions on $\mathbb{Z}$	330
Appendix D: Existence of the Specific Gibbs Free Energy		332
D.1.	Existence Along Hypercubes	332
D.2.	An Extension	334
List of Frequently Used Symbols		335
References		338
Author Index		353
Subject Index		359