

I. Introduction to Homotopy Theory

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Translated from the Russian
by C.J. Shaddock

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II. Homology and Cohomology

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