

## Table of Contents

|   |    |
|---|----|
| Chapter 1. Infinite Galois Theory and Profinite Groups . . . . .      | 1  |
| 1.1 Inverse Limits . . . . .  | 1  |
| 1.2 Profinite Groups . . . . .  | 4  |
| 1.3 Infinite Galois Theory . . . . .                                  | 9  |
| 1.4 The $p$ -adic Integers and the Prüfer Group . . . . .             | 12 |
| 1.5 The Absolute Galois Group of a Finite Field . . . . .             | 15 |
| Exercises . . . . .   | 16 |
| Notes . . . . .   | 18 |
| Chapter 2. Valuations and Linear Disjointness . . . . .               | 19 |
| 2.1 Valuations, Places, and Valuation Rings . . . . .                 | 19 |
| 2.2 Discrete Valuations . . . . .                                     | 21 |
| 2.3 Extensions of Valuations and Places . . . . .                     | 24 |
| 2.4 Integral Extensions and Dedekind Domains . . . . .                | 30 |
| 2.5 Linear Disjointness of Fields . . . . .                           | 34 |
| 2.6 Separable, Regular, and Primary Extensions . . . . .              | 38 |
| 2.7 The Imperfect Degree of a Field . . . . .                         | 44 |
| 2.8 Derivatives . . . . .   | 48 |
| Exercises . . . . .   | 50 |
| Notes . . . . .   | 51 |
| Chapter 3. Algebraic Function Fields of One Variable . . . . .        | 52 |
| 3.1 Function Fields of One Variable . . . . .                         | 52 |
| 3.2 The Riemann-Roch Theorem . . . . .                                | 54 |
| 3.3 Holomorphy Rings . . . . .  | 56 |
| 3.4 Extensions of Function Fields . . . . .                           | 59 |
| 3.5 Completions . . . . .   | 61 |
| 3.6 The Different . . . . .   | 67 |
| 3.7 Hyperelliptic Fields . . . . .                                    | 70 |
| 3.8 Hyperelliptic Fields with a Rational quadratic Subfield . . . . . | 73 |
| Exercises . . . . .   | 75 |
| Notes . . . . .   | 76 |
| Chapter 4. The Riemann Hypothesis for Function Fields . . . . .       | 77 |
| 4.1 Class Numbers . . . . .   | 77 |
| 4.2 Zeta Functions . . . . .  | 79 |
| 4.3 Zeta Functions under Constant Field Extensions . . . . .          | 81 |
| 4.4 The Functional Equation . . . . .                                 | 82 |
| 4.5 The Riemann Hypothesis and Degree 1 Prime Divisors . . . . .      | 84 |
| 4.6 Reduction Steps . . . . .   | 86 |
| 4.7 An Upper Bound . . . . .  | 87 |
| 4.8 A Lower Bound . . . . .   | 89 |

|  |     |
|--|-----|
| Exercises . . . . .  | 91  |
| Notes . . . . .  | 93  |
| <br>   |     |
| Chapter 5. Plane Curves . . . . .                          | 95  |
| 5.1 Affine and Projective Plane Curves . . . . .           | 95  |
| 5.2 Points and prime divisors . . . . .                    | 97  |
| 5.3 The Genus of a Plane Curve . . . . .                   | 99  |
| 5.4 Points on a Curve over a Finite Field . . . . .        | 104 |
| Exercises . . . . .  | 105 |
| Notes . . . . .  | 106 |
| <br>   |     |
| Chapter 6. The Chebotarev Density Theorem . . . . .        | 107 |
| 6.1 Decomposition Groups . . . . .                         | 107 |
| 6.2 The Artin Symbol over Global Fields . . . . .          | 111 |
| 6.3 Dirichlet Density . . . . .                            | 113 |
| 6.4 Function Fields . . . . .                              | 115 |
| 6.5 Number Fields . . . . .                                | 121 |
| Exercises . . . . .  | 129 |
| Notes . . . . .  | 130 |
| <br>   |     |
| Chapter 7. Ultraproducts . . . . .                         | 132 |
| 7.1 First Order Predicate Calculus . . . . .               | 132 |
| 7.2 Structures . . . . .                                   | 134 |
| 7.3 Models . . . . .                                       | 135 |
| 7.4 Elementary Substructures . . . . .                     | 137 |
| 7.5 Ultrafilters . . . . .                                 | 138 |
| 7.6 Regular Ultrafilters . . . . .                         | 139 |
| 7.7 Ultraproducts . . . . .                                | 141 |
| 7.8 Regular Ultraproducts . . . . .                        | 145 |
| 7.9 Nonprincipal Ultraproducts of Finite Fields . . . . .  | 147 |
| Exercises . . . . .  | 147 |
| Notes . . . . .  | 148 |
| <br>   |     |
| Chapter 8. Decision Procedures . . . . .                   | 149 |
| 8.1 Deduction Theory . . . . .                             | 149 |
| 8.2 Gödel's Completeness Theorem . . . . .                 | 152 |
| 8.3 Primitive Recursive Functions . . . . .                | 154 |
| 8.4 Primitive Recursive Relations . . . . .                | 156 |
| 8.5 Recursive Functions . . . . .                          | 157 |
| 8.6 Recursive and Primitive Recursive Procedures . . . . . | 159 |
| 8.7 A Reduction Step in Decidability Procedures . . . . .  | 160 |
| Exercises . . . . .  | 161 |
| Notes . . . . .  | 162 |

|  |     |
|--|-----|
| Chapter 9. Algebraically Closed Fields . . . . .                 | 163 |
| 9.1 Elimination of Quantifiers . . . . .                         | 163 |
| 9.2 A Quantifiers Elimination Procedure . . . . .                | 165 |
| 9.3 Effectiveness . . . . .                                      | 168 |
| 9.4 Applications . . . . .                                       | 169 |
| Exercises . . . . .  | 170 |
| Notes . . . . .  | 170 |
| Chapter 10. Elements of Algebraic Geometry . . . . .             | 172 |
| 10.1 Algebraic Sets . . . . .                                    | 172 |
| 10.2 Varieties . . . . .   | 175 |
| 10.3 Substitutions in Irreducible Polynomials . . . . .          | 176 |
| 10.4 Rational Maps . . . . .                                     | 178 |
| 10.5 Hyperplane Sections . . . . .                               | 180 |
| 10.6 Descent . . . . .   | 182 |
| 10.7 Projective Varieties . . . . .                              | 185 |
| 10.8 About the Language of Algebraic Geometry . . . . .          | 187 |
| Exercises . . . . .  | 190 |
| Notes . . . . .  | 191 |
| Chapter 11. Pseudo Algebraically Closed Fields . . . . .         | 192 |
| 11.1 PAC Fields . . . . .  | 192 |
| 11.2 Reduction to Plane Curves . . . . .                         | 193 |
| 11.3 The PAC Property is an Elementary Statement . . . . .       | 199 |
| 11.4 PAC Fields of Positive Characteristic . . . . .             | 201 |
| 11.5 PAC Fields with Valuations . . . . .                        | 203 |
| 11.6 The Absolute Galois Group of a PAC Field . . . . .          | 207 |
| 11.7 A non-PAC Field $K$ with $K_{\text{ins}}$ PAC . . . . .     | 211 |
| Exercises . . . . .  | 217 |
| Notes . . . . .  | 218 |
| Chapter 12. Hilbertian Fields . . . . .                          | 219 |
| 12.1 Hilbert Sets and Reduction Lemmas . . . . .                 | 219 |
| 12.2 Hilbert Sets under Separable Algebraic Extensions . . . . . | 223 |
| 12.3 Purely Inseparable Extensions . . . . .                     | 224 |
| 12.4 Imperfect fields . . . . .                                  | 228 |
| Exercises . . . . .  | 229 |
| Notes . . . . .  | 230 |
| Chapter 13. The Classical Hilbertian Fields . . . . .            | 231 |
| 13.1 Further Reduction . . . . .                                 | 231 |
| 13.2 Function Fields over Infinite Fields . . . . .              | 236 |
| 13.3 Global Fields . . . . .                                     | 237 |
| 13.4 Hilbertian Rings . . . . .                                  | 241 |
| 13.5 Hilbertianity via Coverings . . . . .                       | 244 |

|   |     |
|---|-----|
| 13.6 Non-Hilbertian $g$ -Hilbertian Fields . . . . .                                    | 248 |
| 13.7 Twisted Wreath Products . . . . .  | 252 |
| 13.8 The Diamond Theorem . . . . .  | 258 |
| 13.9 Weissauer's Theorem . . . . .  | 262 |
| Exercises . . . . .   | 264 |
| Notes . . . . .   | 266 |
| <br>Chapter 14. Nonstandard Structures . . . . .  | 267 |
| 14.1 Higher Order Predicate Calculus . . . . .  | 267 |
| 14.2 Enlargements . . . . .   | 268 |
| 14.3 Concurrent Relations . . . . .   | 270 |
| 14.4 The Existence of Enlargements . . . . .  | 272 |
| 14.5 Examples . . . . .   | 274 |
| Exercises . . . . .   | 275 |
| Notes . . . . .   | 276 |
| <br>Chapter 15. Nonstandard Approach<br>to Hilbert's Irreducibility Theorem . . . . .   | 277 |
| 15.1 Criteria for Hilbertianity . . . . .   | 277 |
| 15.2 Arithmetical Primes Versus Functional Primes . . . . .                             | 279 |
| 15.3 Fields with the Product Formula . . . . .  | 281 |
| 15.4 Generalized Krull Domains . . . . .  | 283 |
| 15.5 Examples . . . . .   | 286 |
| Exercises . . . . .   | 289 |
| Notes . . . . .   | 290 |
| <br>Chapter 16. Galois Groups over Hilbertian Fields . . . . .                          | 291 |
| 16.1 Galois Groups of Polynomials . . . . .   | 291 |
| 16.2 Stable Polynomials . . . . .   | 294 |
| 16.3 Regular Realization of Finite Abelian Groups . . . . .                             | 298 |
| 16.4 Split Embedding Problems with Abelian Kernels . . . . .                            | 302 |
| 16.5 Embedding Quadratic Extensions in $\mathbb{Z}/2^n\mathbb{Z}$ -extensions . . . . . | 306 |
| 16.6 $\mathbb{Z}_p$ -Extensions of Hilbertian Fields . . . . .                          | 308 |
| 16.7 Symmetric and Alternating Groups over Hilbertian Fields . . . . .                  | 315 |
| 16.8 GAR-Realizations . . . . .   | 321 |
| 16.9 Embedding Problems over Hilbertian Fields . . . . .                                | 325 |
| 16.10 Finitely Generated Profinite Groups . . . . .                                     | 328 |
| 16.11 Abelian Extensions of Hilbertian Fields . . . . .                                 | 332 |
| 16.12 Regularity of Finite Groups<br>over Complete Discrete Valued Fields . . . . .     | 334 |
| Exercises . . . . .   | 335 |
| Notes . . . . .   | 336 |
| <br>Chapter 17. Free Profinite Groups . . . . .   | 338 |
| 17.1 The Rank of a Profinite Group . . . . .  | 338 |

|  |     |
|--|-----|
| 17.2 Profinite Completions of Groups . . . . .   | 340 |
| 17.3 Formations of Finite Groups . . . . .   | 344 |
| 17.4 Free pro- $\mathcal{C}$ Groups . . . . .  | 346 |
| 17.5 Subgroups of Free Discrete Groups . . . . .   | 350 |
| 17.6 Open Subgroups of Free Profinite Groups . . . . .                                   | 358 |
| 17.7 An Embedding Property . . . . .   | 360 |
| Exercises . . . . .  | 361 |
| Notes . . . . .  | 362 |
| <br>Chapter 18. The Haar Measure . . . . .   | 363 |
| 18.1 The Haar Measure of a Profinite Group . . . . .                                     | 363 |
| 18.2 Existence of the Haar Measure . . . . .   | 366 |
| 18.3 Independence . . . . .  | 370 |
| 18.4 Cartesian Product of Haar Measures . . . . .  | 376 |
| 18.5 The Haar Measure of the Absolute Galois Group . . . . .                             | 378 |
| 18.6 The PAC Nullstellensatz . . . . .   | 380 |
| 18.7 The Bottom Theorem . . . . .  | 382 |
| 18.8 PAC Fields over Uncountable Hilbertian Fields . . . . .                             | 386 |
| 18.9 On the Stability of Fields . . . . .  | 390 |
| 18.10 PAC Galois Extensions of Hilbertian Fields . . . . .                               | 394 |
| 18.11 Algebraic Groups . . . . .   | 397 |
| Exercises . . . . .  | 400 |
| Notes . . . . .  | 401 |
| <br>Chapter 19. Effective Field Theory and Algebraic Geometry . . . . .                  | 403 |
| 19.1 Presented Rings and Fields . . . . .  | 403 |
| 19.2 Extensions of Presented Fields . . . . .  | 406 |
| 19.3 Galois Extensions of Presented Fields . . . . .                                     | 411 |
| 19.4 The Algebraic and Separable Closures of Presented Fields . . . . .                  | 412 |
| 19.5 Constructive Algebraic Geometry . . . . .   | 413 |
| 19.6 Presented Rings and Constructible Sets . . . . .                                    | 422 |
| 19.7 Basic Normal Stratification . . . . .   | 425 |
| Exercises . . . . .  | 427 |
| Notes . . . . .  | 428 |
| <br>Chapter 20. The Elementary Theory of $e$ -Free PAC Fields . . . . .                  | 429 |
| 20.1 $\aleph_1$ -Saturated PAC Fields . . . . .  | 429 |
| 20.2 The Elementary Equivalence Theorem<br>of $\aleph_1$ -Saturated PAC Fields . . . . . | 430 |
| 20.3 Elementary Equivalence of PAC Fields . . . . .                                      | 433 |
| 20.4 On $e$ -Free PAC Fields . . . . .   | 436 |
| 20.5 The Elementary Theory of Perfect $e$ -Free PAC Fields . . . . .                     | 438 |
| 20.6 The Probable Truth of a Sentence . . . . .  | 440 |
| 20.7 Change of Base Field . . . . .  | 442 |
| 20.8 The Fields $K_s(\sigma_1, \dots, \sigma_e)$ . . . . .                               | 444 |

|  |     |
|--|-----|
| 20.9 The Transfer Theorem . . . . .  | 446 |
| 20.10 The Elementary Theory of Finite Fields . . . . .                     | 448 |
| Exercises . . . . .  | 451 |
| Notes . . . . .  | 453 |
| <br>Chapter 21. Problems of Arithmetical Geometry . . . . .                | 454 |
| 21.1 The Decomposition-Intersection Procedure . . . . .                    | 454 |
| 21.2 $C_i$ -Fields and Weakly $C_i$ -Fields . . . . .                      | 455 |
| 21.3 Perfect PAC Fields which are $C_i$ . . . . .                          | 460 |
| 21.4 The Existential Theory of PAC Fields . . . . .                        | 462 |
| 21.5 Kronecker Classes of Number Fields . . . . .                          | 463 |
| 21.6 Davenport's Problem . . . . .   | 467 |
| 21.7 On permutation Groups . . . . .                                       | 472 |
| 21.8 Schur's Conjecture . . . . .  | 479 |
| 21.9 Generalized Carlitz's Conjecture . . . . .                            | 489 |
| Exercises . . . . .  | 493 |
| Notes . . . . .  | 495 |
| <br>Chapter 22. Projective Groups and Frattini Covers . . . . .            | 497 |
| 22.1 The Frattini Groups of a Profinite Group . . . . .                    | 497 |
| 22.2 Cartesian Squares . . . . .   | 499 |
| 22.3 On $\mathcal{C}$ -Projective Groups . . . . .                         | 502 |
| 22.4 Projective Groups . . . . .   | 506 |
| 22.5 Frattini Covers . . . . .   | 508 |
| 22.6 The Universal Frattini Cover . . . . .                                | 513 |
| 22.7 Projective Pro- $p$ -Groups . . . . .                                 | 515 |
| 22.8 Supernatural Numbers . . . . .  | 520 |
| 22.9 The Sylow Theorems . . . . .  | 522 |
| 22.10 On Complements of Normal Subgroups . . . . .                         | 524 |
| 22.11 The Universal Frattini $p$ -Cover . . . . .                          | 528 |
| 22.12 Examples of Universal Frattini $p$ -Covers . . . . .                 | 532 |
| 22.13 The Special Linear Group $\mathrm{SL}(2, \mathbb{Z}_p)$ . . . . .    | 534 |
| 22.14 The General Linear Group $\mathrm{GL}(2, \mathbb{Z}_p)$ . . . . .    | 537 |
| Exercises . . . . .  | 539 |
| Notes . . . . .  | 542 |
| <br>Chapter 23. PAC Fields and Projective Absolute Galois Groups . . . . . | 544 |
| 23.1 Projective Groups as Absolute Galois Groups . . . . .                 | 544 |
| 23.2 Countably Generated Projective Groups . . . . .                       | 546 |
| 23.3 Perfect PAC Fields of Bounded Corank . . . . .                        | 549 |
| 23.4 Basic Elementary Statements . . . . .                                 | 550 |
| 23.5 Reduction Steps . . . . .   | 554 |
| 23.6 Application of Ultraproducts . . . . .                                | 558 |
| Exercises . . . . .  | 561 |
| Notes . . . . .  | 561 |

|   |     |
|---|-----|
| Chapter 24. Frobenius Fields . . . . .  | 562 |
| 24.1 The Field Crossing Argument . . . . .  | 562 |
| 24.2 The Beckmann-Black Problem . . . . .   | 565 |
| 24.3 The Embedding Property and Maximal Frattini Covers . . . . .                 | 567 |
| 24.4 The Smallest Embedding Cover of a Profinite Group . . . . .                  | 569 |
| 24.5 A Decision Procedure . . . . .   | 574 |
| 24.6 Examples . . . . .   | 576 |
| 24.7 Non-projective Smallest Embedding Cover . . . . .                            | 579 |
| 24.8 A Theorem of Iwasawa . . . . .   | 581 |
| 24.9 Free Profinite Groups of at most Countable Rank . . . . .                    | 583 |
| 24.10 Application of the Nielsen-Schreier Formula . . . . .                       | 586 |
| Exercises . . . . .   | 591 |
| Notes . . . . .   | 592 |
| Chapter 25. Free Profinite Groups of Infinite Rank . . . . .                      | 594 |
| 25.1 Characterization of Free Profinite Groups<br>by Embedding Problems . . . . . | 595 |
| 25.2 Applications of Theorem 25.1.7 . . . . .                                     | 601 |
| 25.3 The Pro- $C$ Completion of a Free Discrete Group . . . . .                   | 604 |
| 25.4 The Group Theoretic Diamond Theorem . . . . .                                | 606 |
| 25.5 The Melnikov Group of a Profinite Group . . . . .                            | 613 |
| 25.6 Homogeneous Pro- $C$ Groups . . . . .  | 615 |
| 25.7 The $S$ -rank of Closed Normal Subgroups . . . . .                           | 620 |
| 25.8 Closed Normal Subgroups with a Basis Element . . . . .                       | 623 |
| 25.9 Accessible Subgroups . . . . .   | 625 |
| Notes . . . . .   | 633 |
| Chapter 26. Random Elements in Free Profinite Groups . . . . .                    | 635 |
| 26.1 Random Elements in a Free Profinite Group . . . . .                          | 635 |
| 26.2 Random Elements in Free pro- $p$ Groups . . . . .                            | 640 |
| 26.3 Random $e$ -tuples in $\hat{\mathbb{Z}}^n$ . . . . .                         | 642 |
| 26.4 On the Index of Normal Subgroups<br>Generated by Random Elements . . . . .   | 646 |
| 26.5 Freeness of Normal Subgroups<br>Generated by Random Elements . . . . .       | 651 |
| Notes . . . . .   | 654 |
| Chapter 27. Omega-Free PAC Fields . . . . .                                       | 655 |
| 27.1 Model Companions . . . . .   | 655 |
| 27.2 The Model Companion in an Augmented Theory of Fields . .                     | 659 |
| 27.3 New Non-Classical Hilbertian Fields . . . . .                                | 664 |
| 27.4 An abundance of $\omega$ -Free PAC Fields . . . . .                          | 667 |
| Notes . . . . .   | 670 |

|   |     |
|---|-----|
| Chapter 28. Undecidability . . . . .  | 671 |
| 28.1 Turing Machines . . . . .  | 671 |
| 28.2 Computation of Functions by Turing Machines . . . . .  | 672 |
| 28.3 Recursive Inseparability of Sets of Turing Machines . . . . .                                | 676 |
| 28.4 The Predicate Calculus . . . . .   | 679 |
| 28.5 Undecidability in the Theory of Graphs . . . . .   | 682 |
| 28.6 Assigning Graphs to Profinite Groups . . . . .   | 687 |
| 28.7 The Graph Conditions . . . . .   | 688 |
| 28.8 Assigning Profinite Groups to Graphs . . . . .   | 690 |
| 28.9 Assigning Fields to Graphs . . . . .   | 694 |
| 28.10 Interpretation of the Theory of Graphs<br>in the Theory of Fields . . . . .                 | 694 |
| Exercises . . . . .   | 697 |
| Notes . . . . .   | 697 |
| <br>  |     |
| Chapter 29. Algebraically Closed Fields with<br>Distinguished Automorphisms . . . . .             | 698 |
| 29.1 The Base Field $K$ . . . . .   | 698 |
| 29.2 Coding in PAC Fields with Monadic Quantifiers . . . . .                                      | 700 |
| 29.3 The Theory of Almost all $\langle \tilde{K}, \sigma_1, \dots, \sigma_e \rangle$ 's . . . . . | 704 |
| 29.4 The Probability of Truth Sentences . . . . .   | 706 |
| <br>  |     |
| Chapter 30. Galois Stratification . . . . .   | 708 |
| 30.1 The Artin Symbol . . . . .   | 708 |
| 30.2 Conjugacy Domains under Projection . . . . .   | 710 |
| 30.3 Normal Stratification . . . . .  | 715 |
| 30.4 Elimination of One Variable . . . . .  | 717 |
| 30.5 The Complete Elimination Procedure . . . . .   | 720 |
| 30.6 Model-Theoretic Applications . . . . .   | 722 |
| 30.7 A Limit of Theories . . . . .  | 725 |
| Exercises . . . . .   | 726 |
| Notes . . . . .   | 729 |
| <br>  |     |
| Chapter 31. Galois Stratification over Finite Fields . . . . .                                    | 730 |
| 31.1 The Elementary Theory of Frobenius Fields . . . . .  | 730 |
| 31.2 The Elementary Theory of Finite Fields . . . . .   | 735 |
| 31.3 Near Rationality of the Zeta Function of a Galois Formula . .                                | 739 |
| Exercises . . . . .   | 748 |
| Notes . . . . .   | 750 |
| <br>  |     |
| Chapter 32. Problems of Field Arithmetic . . . . .  | 751 |
| 32.1 Open Problems of the First Edition . . . . .   | 751 |
| 32.2 Open Problems of the Second Edition . . . . .  | 754 |
| 32.3 Open problems . . . . .  | 758 |

Table of Contents xv

|                      |     |
|----------------------|-----|
| References . . . . . | 761 |
| Index . . . . .      | 780 |