Contents

I Differential Calculus in the Complex Plane $\mathbb{C}$ 9
I.1 Complex Numbers 9
I.2 Convergent Sequences and Series 24
I.3 Continuity 36
I.4 Complex Derivatives 42
I.5 The Cauchy-Riemann Differential Equations 47

II Integral Calculus in the Complex Plane $\mathbb{C}$ 69
II.1 Complex Line Integrals 70
II.2 The Cauchy Integral Theorem 77
II.3 The Cauchy Integral Formulas 92

III Sequences and Series of Analytic Functions, the Residue Theorem 103
III.1 Uniform Approximation 104
III.2 Power Series 109
III.3 Mapping Properties of Analytic Functions 124
III.4 Singularities of Analytic Functions 133
III.5 Laurent Decomposition 142
A Appendix to III.4 and III.5 155
III.6 The Residue Theorem 162
III.7 Applications of the Residue Theorem 170

IV Construction of Analytic Functions 191
IV.1 The Gamma Function 192
IV.2 The Weierstrass Product Formula 210
IV.3 The Mittag-Leffler Partial Fraction Decomposition 218
IV.4 The Riemann Mapping Theorem 223
A Appendix : The Homotopical Version of the Cauchy Integral Theorem 233
B Appendix : A Homological Version of the Cauchy Integral Theorem 239

C Appendix : Characterizations of Elementary Domains 244

V Elliptic Functions 251
V.1 Liouville's Theorems 252
A Appendix to the Definition of the Period Lattice 259
V.2 The Weierstrass $\wp$-function 261
V.3 The Field of Elliptic Functions 267
A Appendix to Sect. V.3 : The Torus as an Algebraic Curve 271
V.4 The Addition Theorem 278
V.5 Elliptic Integrals 284
V.6 Abel's Theorem 291
V.7 The Elliptic Modular Group 301
V.8 The Modular Function j 309

VI Elliptic Modular Forms 317
VI.1 The Modular Group and Its Fundamental Region 318
VI.2 The $k/12$-formula and the Injectivity of the j-function 326
VI.3 The Algebra of Modular Forms 334
VI.4 Modular Forms and Theta Series 338
VI.5 Modular Forms for Congruence Groups 352
A Appendix to VI.5 : The Theta Group 363
VI.6 A Ring of Theta Functions 370

VII Analytic Number Theory 381
VII.1 Sums of Four and Eight Squares 382
VII.2 Dirichlet Series 399
VII.3 Dirichlet Series with Functional Equations 408
VII.4 The Riemann ζ-function and Prime Numbers 421
VII.5 The Analytic Continuation of the ζ-function 429
VII.6 A Tauberian Theorem 436

VIII Solutions to the Exercises 449
VIII.1 Solutions to the Exercises of Chapter I 449
VIII.2 Solutions to the Exercises of Chapter II 459
VIII.3 Solutions to the Exercises of Chapter III 464
VIII.4 Solutions to the Exercises of Chapter IV 475
VIII.5 Solutions to the Exercises of Chapter V 482
VIII.6 Solutions to the Exercises of Chapter VI 490
VIII.7 Solutions to the Exercises of Chapter VII 498

References 509

Symbolic Notations 519